Jónsson Properties for Non-Ordinal Sets Under the Axiom of Determinacy

Jared Holshouser and Stephen Jackson

University of North Texas

Fifth NY Graduate Student Logic Conference

The Jónsson Property

Definition

For κ a cardinal and $n \in \omega$,
$[\kappa]^n = \{(\alpha_1, \cdots, \alpha_n) \in \kappa^n : \alpha_1 < \cdots < \alpha_n\}$. We also set
$[\kappa]^{<\omega} = \bigcup_{n\in\omega}[\kappa]^n$.

Definition

We say that κ is **Jónsson** iff whenever $f : [\kappa]^{<\omega} \to \kappa$, there is an $H \subseteq \kappa$ so that $|H| = \kappa$ and $f[[H]^{<\omega}] \neq \kappa$.

The Jónsson Property

Definition

For κ a cardinal and $n \in \omega$,
$[\kappa]^n = \{(\alpha_1, \cdots, \alpha_n) \in \kappa^n : \alpha_1 < \cdots < \alpha_n\}$. We also set
$[\kappa]^{<\omega} = \bigcup_{n\in\omega} [\kappa]^n$.

Definition

We say that κ is **Jónsson** iff whenever $f : [\kappa]^{<\omega} \to \kappa$, there is an $H \subseteq \kappa$ so that $|H| = \kappa$ and $f[[H]^{<\omega}] \neq \kappa$.

Remark

In ZFC, the existence of a Jónsson cardinal implies the existence of $0^\#$ and is implied by the existence of a measurable cardinal [4].

Some AD Notions

Definition

Recall that under AD, $\mathbb{R}$ cannot be well-ordered. We define Θ to be least cardinal that $\mathbb{R}$ does not surject onto.

Definition

Recall that $L(\mathbb{R})$ is the minimal universe of ZF which contains $\mathbb{R}$. Under large cardinal hypotheses, $L(\mathbb{R})$ is a model of AD, and its theory is absolute for very complex statements.

Some AD Notions

Definition

Recall that under AD, $\mathbb{R}$ cannot be well-ordered. We define Θ to be least cardinal that $\mathbb{R}$ does not surject onto.

Definition

Recall that $L(\mathbb{R})$ is the minimal universe of ZF which contains $\mathbb{R}$. Under large cardinal hypotheses, $L(\mathbb{R})$ is a model of AD, and its theory is absolute for very complex statements.

Remark

It has been shown that under AD, ordinary cardinals have large cardinal properties in $L(\mathbb{R})$. For instance, ω_1 is a measurable cardinal.

The Jónsson Property Under AD

In 2015, S. Jackson, R. Ketchersid, F. Schlutzenberg, and W.H. Woodin [3] proved the following:

Theorem ($AD + V = L(\mathbb{R})$, J/K/S/W)

Let $\kappa < \Theta$ be an uncountable cardinal. Then κ is Jónsson. In fact, if λ is a cardinal between ω_1 and κ, and $f : [\kappa]^{<\omega} \to \lambda$, then there is an $H \subseteq \kappa$ so that $|H| = \kappa$ and

$$|\lambda - f[[H]^{<\omega}]| = \lambda.$$

The Jónsson Property Under AD

In 2015, S. Jackson, R. Ketchersid, F. Schlutzenberg, and W.H. Woodin [3] proved the following:

Theorem (AD + $V = L(\mathbb{R})$, J/K/S/W)

Let $\kappa < \Theta$ be an uncountable cardinal. Then κ is Jónsson. In fact, if λ is a cardinal between ω_1 and κ, and $f : [\kappa]^{<\omega} \to \lambda$, then there is an $H \subseteq \kappa$ so that $|H| = \kappa$ and

$$|\lambda - f[[H]^{<\omega}]| = \lambda.$$

In this paper, they asked whether or not there were non-ordinal Jónsson cardinals. In particular, is $\mathbb{R}$ Jónsson?

Reframing the Question

Definition

For any set A, $[A]^n = \{s \subseteq X : |s| = n\}$ and $[A]^{<\omega} = \bigcup_{n\in\omega}[A]^n$.

Definition

Let A and B be infinite sets.

- A is **Jónsson** iff for any $f : [A]^{<\omega} \to A$, there is an $X \subseteq A$ so that $|X| = |A|$ and $f[[X]^{<\omega}] \neq A$.
- (A, B) is a **Jónsson pair** iff for any $f : [A]^{<\omega} \to B$, there is an $X \subseteq A$ so that $|X| = |A|$ and $f[[X]^{<\omega}] \neq B$.
- A is **strongly Jónsson** iff for any $f : [A]^{<\omega} \to A$, there is an $X \subseteq A$ so that $|X| = |A|$ and
$$|A - f[[X]^{<\omega}]| = |A|.$$
- (A, B) is a **strong Jónsson pair** iff for any $f : [A]^{<\omega} \to B$, there is an $X \subseteq A$ so that $|X| = |A|$ and
$$|B - f[[X]^{<\omega}]| = |B|.$$

Tools From Descriptive Set Theory

We use the following repeatedly.

Lemma (Fusion Lemma)

For each $s \in 2^{<\omega}$ let P_s be a perfect set so that

1. $\lim_{|s|\to\infty} diam(P_s) = 0$, *and*
2. *for all $s \in 2^{<\omega}$, $P_{s\frown 0} \cap P_{s\frown 1} = \emptyset$ and $P_{s\frown 0}, P_{s\frown 1} \subsetneq P_s$.*

Then the fusion $P = \bigcup_{f\in 2^\omega} \bigcap_{n\in\omega} P_{f|_n}$ of $\langle P_s : s \in 2^{<\omega}\rangle$ is a perfect set.

Theorem (Mycielski)

Suppose $C_n \subseteq (2^\omega)^n$ are comeager for all $n \in \omega$. Then there is a perfect set $P \subseteq 2^\omega$ so that $[P]^n \subseteq C_n$ for all n.

$\mathbb{R}$ is Strongly Jónsson

Theorem (AD, H./Jackson)

$\mathbb{R}$ *is Strongly Jónsson.*

$\mathbb{R}$ is Strongly Jónsson

Theorem (AD, H./Jackson)

$\mathbb{R}$ is Strongly Jónsson.

Proof.

- We can break f into component functions, f_n.
- Find comeager sets on which the f_n are continuous.
- Use the result of Mycielski[5] to thread a perfect set through the comeager sets.
- Use continuity and the fusion lemma to inductively thin out the range of the f_n.

□

$\mathbb{R}$ and Cardinals

Proposition (AD, H./Jackson)

If $\kappa < \Theta$ is an uncountable cardinal, then $(\mathbb{R}, \kappa)$ and $(\kappa, \mathbb{R})$ are Rowbottom ((A, B) is **Rowbottom** *iff whenever $f : [A]^{<\omega} \to B$ there is an $X \subseteq A$ so that $|X| = |A|$ and $f[[X]^{<\omega}]$ is countable.)*

Proposition (AD, H./Jackson)

Let $\kappa, \lambda < \Theta$ be uncountable cardinals. Let $\odot, *$ be $\cup$ or x. Then $(\kappa \odot \mathbb{R}, \lambda * \mathbb{R})$ is a strong Jónsson pair.

$\mathbb{R}$ and Cardinals

Proposition (AD, H./Jackson)

If $\kappa < \Theta$ is an uncountable cardinal, then $(\mathbb{R}, \kappa)$ and $(\kappa, \mathbb{R})$ are Rowbottom ((A, B) is **Rowbottom** *iff whenever $f : [A]^{<\omega} \to B$ there is an $X \subseteq A$ so that $|X| = |A|$ and $f[[X]^{<\omega}]$ is countable.)*

Proposition (AD, H./Jackson)

Let $\kappa, \lambda < \Theta$ be uncountable cardinals. Let $\odot, *$ be $\cup$ or x. Then $(\kappa \odot \mathbb{R}, \lambda * \mathbb{R})$ is a strong Jónsson pair.

What about other non-ordinal sets?

Jónsson Properties for General Sets

Suppose $X \in L_\Theta(\mathbb{R})$. Then there is a surjection $F : \mathbb{R} \to X$. We can define an equivalence relation E on $\mathbb{R}$ by

$$xEy \iff F(x) = F(y).$$

Jónsson Properties for General Sets

Suppose $X \in L_\Theta(\mathbb{R})$. Then there is a surjection $F : \mathbb{R} \to X$. We can define an equivalence relation E on $\mathbb{R}$ by

$$xEy \iff F(x) = F(y).$$

Note that X is in bijection with $\mathbb{R}/E$. There is then a (possibly not unique) decomposition of $\mathbb{R}/E$ into a well-ordered component and another component which $\mathbb{R}$ surjects onto and injects into [2]. Call the surjection ϕ^X and the injection ϕ_X. Either of these components could be empty.

Jónsson Properties for General Sets

Suppose $X \in L_\Theta(\mathbb{R})$. Then there is a surjection $F : \mathbb{R} \to X$. We can define an equivalence relation E on $\mathbb{R}$ by

$$xEy \iff F(x) = F(y).$$

Note that X is in bijection with $\mathbb{R}/E$. There is then a (possibly not unique) decomposition of $\mathbb{R}/E$ into a well-ordered component and another component which $\mathbb{R}$ surjects onto and injects into [2]. Call the surjection ϕ^X and the injection ϕ_X. Either of these components could be empty.
The most general result currently obtainable is the following:

Theorem (AD + $V = L(\mathbb{R})$, H./Jackson)

Suppose that $X \in L_\Theta(\mathbb{R})$ is in bijection with $\kappa \cup A$, where κ is an uncountable cardinal and $\mathbb{R}$ maps onto and into A. Similarly, suppose $Y \in L_\Theta(\mathbb{R})$ is in bijection with $\lambda \cup B$. Let $f : [\kappa \cup A]^{<\omega} \to \lambda \cup B$. Then there are perfect $P, Q \subseteq \mathbb{R}$ and there is an $H \subseteq \kappa$ with $|H| = \kappa$ so that

$$|\lambda - f[[H \cup \phi^A[P]]^{<\omega}]| = \lambda \text{ and } f[[H \cup \phi^A[P]]^{<\omega}] \cap \phi_B[Q] = \emptyset.$$

Background for E_0

Recall the following:

Definition

Let $x, y \in 2^\omega$. Then xE_0y iff $(\exists N)(\forall n \geq N)[x(n) = y(n)]$.

Note that $2^\omega/E_0$ has no definable linear ordering and E_0 has no definable transversal.

Background for E_0

Recall the following:

Definition

Let $x, y \in 2^\omega$. Then xE_0y iff $(\exists N)(\forall n \geq N)[x(n) = y(n)]$.

Note that $2^\omega/E_0$ has no definable linear ordering and E_0 has no definable transversal.

The following is a corollary of the Glimm-Effros Dichotomy [1]:

Corollary (AD)

Suppose $H \subseteq 2^\omega/E_0$. Then H satisfies exactly one of the following:

- *H is countable,*
- *H is in bijection with $\mathbb{R}$, or*
- *H is in bijection with $2^\omega/E_0$.*

Mycielski for E_0

Definition

$A \subseteq 2^\omega$ has **power $\mathbf{E_0}$** iff A is E_0-saturated and A/E_0 is in bijection with $2^\omega/E_0$.

Definition

For $n \in \omega$ and $A \subseteq 2^\omega$, let

$$[A]^n_{E_0} = \{\vec{x} \in [A]^n : |\{[x_1]_{E_0}, \cdots, [x_n]_{E_0}\}| = n\}$$

Mycielski for E_0

Definition

$A \subseteq 2^\omega$ has **power $\mathbf{E_0}$** iff A is E_0-saturated and A/E_0 is in bijection with $2^\omega/E_0$.

Definition

For $n \in \omega$ and $A \subseteq 2^\omega$, let

$$[A]^n_{E_0} = \{\vec{x} \in [A]^n : |\{[x_1]_{E_0}, \cdots, [x_n]_{E_0}\}| = n\}$$

We were able to prove the following Mycielski style result.

Theorem (H./Jackson)

Suppose that $C_n \subseteq (2^\omega)^n$ are comeager and E_0-saturated for all $n \in \omega$. Then there is an $A \subseteq 2^\omega$ of power E_0 so that $[A]^n_{E_0} \subseteq C_n$ for all n.

$\mathbb{R}/E_0$ is Strongly Jónsson

Theorem (AD, H./Jackson)

$2^\omega/E_0$ is strongly Jónsson.

$\mathbb{R}/E_0$ is Strongly Jónsson

Theorem (AD, H./Jackson)

$2^\omega/E_0$ is strongly Jónsson.

Proof.

- We can lift $f : [2^\omega/E_0]^{<\omega} \to 2^\omega/E_0$ to a function $F : [2^\omega]^{<\omega} \to 2^\omega$ so that

$$\vec{a} E_0 \vec{b} \iff F(\vec{a}) \in f(\{[b_1]_{E_0}, \cdots, [b_n]_{E_0}\}).$$

- We can break F into component functions, F_n.
- Find comeager sets on which the F_n are continuous.
- Use the Mycielski-style result for E_0 to thread a power E_0 set through the comeager sets.
- Use continuity and the techniques of the Mycielski-style result to inductively thin out the range of the F_n.

□

Other Combinations

Proposition (AD, H./Jackson)

Suppose $\kappa, \lambda < \Theta$ are cardinals, $A, B \in \{\kappa, \lambda, \mathbb{R}, 2^\omega/E_0\}$ and $\odot$, $*$ are $\cup$ or $\times$. Then $(\kappa \odot A, \lambda * B)$ is a strong Jónsson pair.

Other Combinations

Proposition (AD, H./Jackson)

Suppose $\kappa, \lambda < \Theta$ are cardinals, $A, B \in \{\kappa, \lambda, \mathbb{R}, 2^\omega/E_0\}$ and $\odot$, $*$ are $\cup$ or $\times$. Then $(\kappa \odot A, \lambda * B)$ is a strong Jónsson pair.

Of particular note is the following:

Proposition (AD, H./Jackson)

Let $\kappa < \Theta$ be an uncountable cardinal. Then

- $(2^\omega/E_0, \mathbb{R})$ *is Ramsey,*
- $(2^\omega/E_0, \kappa)$ *is Ramsey, and*
- $(\kappa, 2^\omega/E_0)$ *is Rowbottom.*

Further Work

- Can the result be extended to well-ordered unions of hyperfinite quotients of $\mathbb{R}$?
- Do infinite partition properties hold for $2^\omega/E_0$?
- Can we get this Mycielski style result for other equivalence relations?
- Can the full Jónsson result be proved for general equivalence relations?

Thanks For Listening!

References I

[1] S. Gao.
Invariant Descriptive Set Theory.
CRC Press, 2008.

[2] L. Harrington and S. Shelah.
Counting equivalence classes for co-κ-suslin equivalence relations.
Logic Colloquium, 108, 1980.

[3] S. Jackson, R. Ketchersid, F. Schlutzenberg, and W. Woodin.
Ad and Jónsson cardinals in $L(\mathbb{R})$.
Journal of Symbolic Logic, 79, 2014.

References II

[4] A. Kanamori.
The Higher Infinite.
Springer, 2009.

[5] J. Mycielski.
Independent sets in topological algebras.
Fund. Math., 55, 1964.